Solution toTheoretical Question 3

Part A

Neutrino Mass and Neutron Decay

(a) Let $(c^2E_e, c\vec{q}_e)$, $(c^2E_p, c\vec{q}_p)$, and $(c^2E_\nu, c\vec{q}_\nu)$ be the energy-momentum 4-vectors of the electron, the proton, and the anti-neutrino, respectively, in the rest frame of the neutron. Notice that $E_e, E_p, E_\nu, \vec{q}_e, \vec{q}_p, \vec{q}_\nu$ are all in units of mass. The proton and the anti-neutrino may be considered as forming a system of total rest mass M_c, total energy c^2E_c, and total momentum $c\vec{q}_c$. Thus, we have

$$E_c = E_p + E_\nu, \qquad \vec{q}_c = \vec{q}_p + \vec{q}_\nu, \qquad M_c^2 = E_c^2 - q_c^2 \tag{A1}$$

Note that the magnitude of the vector $\vec{q}_c$ is denoted as q_c. The same convention also applies to all other vectors.

Since energy and momentum are conserved in the neutron decay, we have

$$E_c + E_e = m_n \tag{A2}$$

$$\vec{q}_c = -\vec{q}_e \tag{A3}$$

When squared, the last equation leads to the following equality

$$q_c^2 = q_e^2 = E_e^2 - m_e^2 \tag{A4}$$

From Eq. (A4) and the third equality of Eq. (A1), we obtain

$$E_c^2 - M_c^2 = E_e^2 - m_e^2 \tag{A5}$$

With its second and third terms moved to the other side of the equality, Eq. (A5) may be divided by Eq. (A2) to give

$$E_c - E_e = \frac{1}{m_n}(M_c^2 - m_e^2) \tag{A6}$$

As a system of coupled linear equations, Eqs. (A2) and (A6) may be solved to give

$$E_c = \frac{1}{2m_n}(m_n^2 - m_e^2 + M_c^2) \tag{A7}$$

$$E_e = \frac{1}{2m_n}(m_n^2 + m_e^2 - M_c^2) \tag{A8}$$

Using Eq. (A8), the last equality in Eq. (A4) may be rewritten as

$$\begin{aligned} q_e &= \frac{1}{2m_n}\sqrt{(m_n^2 + m_e^2 - M_c^2)^2 - (2m_n m_e)^2} \\ &= \frac{1}{2m_n}\sqrt{(m_n + m_e + M_c)(m_n + m_e - M_c)(m_n - m_e + M_c)(m_n - m_e - M_c)} \end{aligned} \tag{A9}$$

Eq. (A8) shows that a maximum of E_e corresponds to a minimum of M_c^2. Now the rest mass M_c is the total energy of the proton and anti-neutrino pair in their center of mass (or momentum) frame so that it achieves the minimum

$$(M_c)_{\min} = M = m_p + m_v \tag{A10}$$

when the proton and the anti-neutrino are both at rest in the center of mass frame. Hence, from Eqs. (A8) and (A10), the maximum energy of the electron $E = c^2 E_e$ is

$$E_{\max} = \frac{c^2}{2m_n}\left[m_n^2 + m_e^2 - (m_p + m_v)^2\right] \approx 1.292569 \text{ MeV} \approx 1.29\,\text{MeV} \tag{A11}$$

When Eq. (A10) holds, the proton and the anti-neutrino move with the same velocity v_m of the center of mass and we have

$$\frac{v_m}{c} = \left(\frac{q_v}{E_v}\right)_{E=E_{\max}} = \left(\frac{q_p}{E_p}\right)_{E=E_{\max}} = \left(\frac{q_c}{E_c}\right)_{E=E_{\max}} = \left(\frac{q_e}{E_c}\right)_{M_c=m_p+m_v} \tag{A12}$$

where the last equality follows from Eq. (A3). By Eqs. (A7) and (A9), the last expression in Eq. (A12) may be used to obtain the speed of the anti-neutrino when $E = E_{\max}$. Thus, with $M = m_p + m_v$, we have

$$\frac{v_m}{c} = \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \tag{A13}$$
$$\approx 0.00126538 \approx 0.00127$$

[Alternative Solution]

Assume that, in the rest frame of the neutron, the electron comes out with momentum $c\vec{q}_e$ and energy $c^2 E_e$, the proton with $c\vec{q}_p$ and $c^2 E_p$, and the anti-neutrino with $c\vec{q}_v$ and $c^2 E_v$. With the magnitude of vector $\vec{q}_\alpha$ denoted by the symbol q_α, we have

$$E_p^2 = m_p^2 + q_p^2, \qquad E_v^2 = m_v^2 + q_v^2, \qquad E_e^2 = m_e^2 + q_e^2 \tag{1A}$$

Conservation of energy and momentum in the neutron decay leads to

$$E_p + E_v = m_n - E_e \tag{2A}$$

$$\vec{q}_p + \vec{q}_v = -\vec{q}_e \tag{3A}$$

When squared, the last two equations lead to

$$E_p^2 + E_v^2 + 2E_p E_v = (m_n - E_e)^2 \tag{4A}$$

$$q_p^2 + q_v^2 + 2\vec{q}_p \cdot \vec{q}_v = q_e^2 = E_e^2 - m_e^2 \tag{5A}$$

Subtracting Eq. (5A) from Eq. (4A) and making use of Eq. (1A) then gives

$$m_p^2 + m_v^2 + 2(E_p E_v - \vec{q}_p \cdot \vec{q}_v) = m_n^2 + m_e^2 - 2m_n E_e \tag{6A}$$

or, equivalently,

$$2m_n E_e = m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2(E_p E_\nu - \vec{q}_p \cdot \vec{q}_\nu) \qquad (7A)$$

If θ is the angle between $\vec{q}_p$ and $\vec{q}_\nu$, we have $\vec{q}_p \cdot \vec{q}_\nu = q_p q_\nu \cos\theta \le q_p q_\nu$ so that Eq. (7A) leads to the relation

$$2m_n E_e \le m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2(E_p E_\nu - q_p q_\nu) \qquad (8A)$$

Note that the equality in Eq. (8A) holds only if $\theta = 0$, i.e., the energy of the electron $c^2 E_e$ takes on its maximum value only when the anti-neutrino and the proton *move in the same direction*.

Let the speeds of the proton and the anti-neutrino in the rest frame of the neutron be $c\beta_p$ and $c\beta_\nu$, respectively. We then have $q_p = \beta_p E_p$ and $q_\nu = \beta_\nu E_\nu$. As shown in Fig. A1, we introduce the angle ϕ_ν ($0 \le \phi_\nu < \pi/2$) for the antineutrino by

$$q_\nu = m_\nu \tan\phi_\nu, \qquad E_\nu = \sqrt{m_\nu^2 + q_\nu^2} = m_\nu \sec\phi_\nu, \qquad \beta_\nu = q_\nu / E_\nu = \sin\phi_\nu \qquad (9A)$$

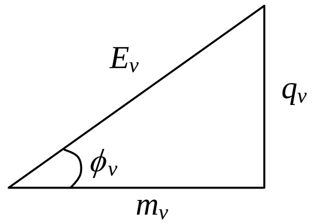

Figure A1

Similarly, for the proton, we write, with $0 \le \phi_p < \pi/2$,

$$q_p = m_p \tan\phi_p, \quad E_p = \sqrt{m_p^2 + q_p^2} = m_p \sec\phi_p, \quad \beta_p = q_p / E_p = \sin\phi_p \qquad (10A)$$

Eq. (8A) may then be expressed as

$$2m_n E_e \le m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2m_p m_\nu \left(\frac{1 - \sin\phi_p \sin\phi_\nu}{\cos\phi_p \cos\phi_\nu} \right) \qquad (11A)$$

The factor in parentheses at the end of the last equation may be expressed as

$$\frac{1 - \sin\phi_p \sin\phi_\nu}{\cos\phi_p \cos\phi_\nu} = \frac{1 - \sin\phi_p \sin\phi_\nu - \cos\phi_p \cos\phi_\nu}{\cos\phi_p \cos\phi_\nu} + 1 = \frac{1 - \cos(\phi_p - \phi_\nu)}{\cos\phi_p \cos\phi_\nu} + 1 \ge 1 \qquad (12A)$$

and clearly assumes its minimum possible value of 1 when $\phi_p = \phi_\nu$, i.e., when the anti-neutrino and the proton *move with the same velocity* so that $\beta_p = \beta_\nu$. Thus, it follows from Eq. (11A) that the maximum value of E_e is

$$\begin{aligned} (E_e)_{\max} &= \frac{1}{2m_n}(m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2m_p m_\nu) \\ &= \frac{1}{2m_n}\left[m_n^2 + m_e^2 - (m_p + m_\nu)^2\right] \end{aligned} \qquad (13A)$$

and the maximum energy of the electron $E = c^2 E_e$ is

$$E_{\max} = c^2 (E_e)_{\max} \approx 1.292569 \text{ MeV} \approx 1.29 \text{ MeV} \qquad (14A)$$

When the anti-neutrino and the proton move with the same velocity, we have, from Eqs. (9A), (10A), (2A) ,(3A), and (1A), the result

$$\beta_\nu = \beta_p = \frac{q_p}{E_p} = \frac{q_\nu}{E_\nu} = \frac{q_p + q_\nu}{E_p + E_\nu} = \frac{q_e}{m_n - E_e} = \frac{\sqrt{E_e^2 - m_e^2}}{m_n - E_e} \tag{15A}$$

Substituting the result of Eq. (13A) into the last equation, the speed v_m of the anti-neutrino when the electron attains its maximum value $E_{\max}$ is, with $M = m_p + m_\nu$, given by

$$\frac{v_m}{c} = (\beta_\nu)_{\max E_e} = \frac{\sqrt{(E_e)_{\max}^2 - m_e^2}}{m_n - (E_e)_{\max}} = \frac{\sqrt{(m_n^2 + m_e^2 - M^2)^2 - 4m_n^2 m_e^2}}{2m_n^2 - (m_n^2 + m_e^2 - M^2)}$$

$$= \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \tag{16A}$$

$$\approx 0.00126538 \approx 0.00127$$

Part B

Light Levitation

(b) Refer to Fig. B1. Refraction of light at the spherical surface obeys Snell's law and leads to

$$n \sin\theta_i = \sin\theta_t \tag{B1}$$

Neglecting terms of the order $(\delta/R)^3$or higher in sine functions, Eq. (B1) becomes

$$n\theta_i \approx \theta_t \tag{B2}$$

For the triangle ΔFAC in Fig. B1, we have

$$\beta = \theta_t - \theta_i \approx n\theta_i - \theta_i = (n-1)\theta_i \tag{B3}$$

Let f_0 be the frequency of the incident light. If n_p is the number of photons incident on the plane surface per unit area per unit time, then the total number of photons incident on the plane surface per unit time is $n_p \pi \delta^2$. The total power P of photons incident on the plane surface is $(n_p \pi \delta^2)(hf_0)$, with h being Planck's constant. Hence,

$$n_p = \frac{P}{\pi \delta^2 h f_0} \tag{B4}$$

The number of photons incident on an annular disk of inner radius r and outer radius $r + dr$ on the plane surface per unit time is $n_p(2\pi r dr)$, where $r = R\tan\theta_i \approx R\theta_i$. Therefore,

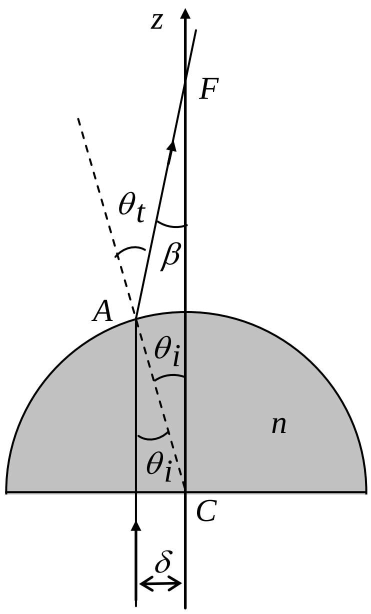

Fig. B1

$$n_p(2\pi r dr) \approx n_p(2\pi R^2)\theta_i d\theta_i \tag{B5}$$

The z-component of the momentum carried away per unit time by these photons when refracted at the spherical surface is

$$
\begin{aligned}
dF_z &= n_p \frac{hf_o}{c}(2\pi r dr)\cos\beta \approx n_p \frac{hf_0}{c}(2\pi R^2)\left(1-\frac{\beta^2}{2}\right)\theta_i d\theta_i \\
&\approx n_p \frac{hf_0}{c}(2\pi R^2)\left[\theta_i - \frac{(n-1)^2}{2}\theta_i^3\right]d\theta_i
\end{aligned}
\tag{B6}
$$

so that the *z*-component of the total momentum carried away per unit time is

$$
\begin{aligned}
F_z &= 2\pi R^2 n_p \left(\frac{hf_0}{c}\right)\int_0^{\theta_{im}}\left[\theta_i - \frac{(n-1)^2}{2}\theta_i^3\right]d\theta_i \\
&= \pi R^2 n_p \left(\frac{hf_0}{c}\right)\theta_{im}^2\left[1-\frac{(n-1)^2}{4}\theta_{im}^2\right]
\end{aligned}
\tag{B7}
$$

where $\tan\theta_{im} = \frac{\delta}{R} \approx \theta_{im}$. Therefore, by the result of Eq. (B5), we have

$$
F_z = \frac{\pi R^2 P}{\pi\delta^2 hf_0}\left(\frac{hf_0}{c}\right)\frac{\delta^2}{R^2}\left[1-\frac{(n-1)^2\delta^2}{4R^2}\right] = \frac{P}{c}\left[1-\frac{(n-1)^2\delta^2}{4R^2}\right]
\tag{B8}
$$

The force of optical levitation is equal to the sum of the *z*-components of the forces exerted by the incident and refracted lights on the glass hemisphere and is given by

$$
\frac{P}{c} + (-F_z) = \frac{P}{c} - \frac{P}{c}\left[1-\frac{(n-1)^2\delta^2}{4R^2}\right] = \frac{(n-1)^2\delta^2}{4R^2}\frac{P}{c}
\tag{B9}
$$

Equating this to the weight *mg* of the glass hemisphere, we obtain the minimum laser power required to levitate the hemisphere as

$$
P = \frac{4mgcR^2}{(n-1)^2\delta^2}
\tag{B10}
$$

Marking Scheme

Theoretical Question 3
Neutrino Mass and Neutron Decay

Total Scores	Sub Scores	Marking Scheme for Answers to the Problem
Part A 4.0 pts.	(a) 4.0	The maximum energy of the electron and the corresponding speed of the anti-neutrino. ➢ 0.5 use energy-momentum conservation and can convert it into equations. ➢ 0.5 obtain an expression for E_e that allows the determination of its maximum value. ➢ (0.5+0.2) for concluding that proton and anti-neutrino must move with the same velocity when E_e is maximum. (0.2 for the same direction) ➢ 0.6 for establishing the minimum value of $(E_p E_\nu - \vec{q}_p \cdot \vec{q}_\nu)$ to be $m_p m_\nu$ or a conclusion equivalent to it. ➢ (0.5+0.1) for expression and value of E_{max}. ➢ 0.5 for concluding $\beta_\nu = \sqrt{E_e^2 - m_e^2}/(m_n - E_e)$. ➢ (0.5+0.1) for expression and value of v_m/c.

Light Levitation

Part B 4.0 pts	(b) 4.0	Laser power needed to balance the weight of the glass hemisphere. ➢ 0.3 for law of refraction $n \sin\theta_i = \sin\theta_t$. ➢ 0.3 for making the linear approximation $n\theta_i \approx \theta_t$. ➢ 0.4 for relation between angles of deviation and incidence. ➢ 0.3 for photon energy $\varepsilon = h\nu$. ➢ 0.3 for photon momentum $p = \varepsilon/c$. ➢ 0.3 for momentum of incident photons per unit time = P/c. ➢ 0.6 for momentum of photons refracted per unit time as a function of the angle of incidence. ➢ 0.4 for total momentum of photons refracted per unit time = $[1-(n-1)^2\delta^2/(4R^2)]P/c$. ➢ 0.4 for force of levitation = sum of forces exerted by incident and refracted photons. ➢ 0.4 for force of levitation = $(n-1)^2\delta^2 P/(4cR^2)$. ➢ 0.3 for the needed laser power $P = 4mgcR^2/(n-1)^2\delta^2$.